Network Quantile Autoregression

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Abstract

The complex tail dependency structure in a dynamic network with a large number of nodes is an important object to study. Here we propose a network quantile autoregression model (NQAR), which characterizes the dynamic quantile behavior. Our NQAR model consists of a system of equations, of which we relate a response to its connected nodes and node specific characteristics in a quantile autoregression process. We show the estimation of the NQAR model and the asymptotic properties with assumptions on the network structure. For this propose we develop a network Bahadur representation that gives us direct insight into the parameter asymptotics. Moreover, innovative tail-event driven impulse functions are defined. Finally, we demonstrate the usage of our model by investigating the financial contagions in the Chinese stock market accounting for shared ownership of companies. We find higher network dependency when the market is exposed to a higher volatility level.

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1. INTRODUCTION

Quantile regression is an increasingly popular tool for modern econometric analysis. Instead of studying the conditional mean function of the response variable, quantile regression is concerned with estimating the conditional quantile function. It has been applied to a wide range of econometric applications, such as labor economics (Koenker and Hallock, 2001; Fitzenberger et al., 2013) and financial risk management (Gaglianone et al., 2011; Härdle et al., 2016). Particularly, the linear quantile model has been studied by a seminal work by Koenker and Bassett (1978), and the asymptotic theory has been developed by Portnoy (1991, 1997). See Koenker (2005) for a comprehensive summary of the methods and applications.

Following the development of quantile methods in the existing literature, the quantile regression in time series is of particular interest. An early stream of researches such as Hasan and Koenker (1997) deal with linear quantile autoregression models, which focus on independent identically distributed (*iid*) innovations in a relatively restrictive location shift model. In another approach, Engle and Manganelli (2004) propose a set of autoregressive forms (the CaViaR model) for value-at-risk (VaR), which models directly the dynamics of the conditional quantiles. The framework is easy to apply but is quite difficult to directly infer the underlying process. As an alternative, Koenker and Xiao (2006) consider a quantile autoregressive method to model the conditional quantile function, which allows to study the asymptotic properties of the underlying process and does not assume an *iid* underlying process. This provides us with an interesting framework to understand the risk propagation within a complex financial system. We consider therefore a parametric approach involving a system of dynamic quantile autoregression equations. Thus our methodology characterizes a dynamic tail dependency graph, which facilitates tail event driven forecasting and impulse analysis in a complex financial system. This is in particular important in a financial network and complements to the literature on systemic risk; see for example Billio et al. (2012), Diebold and Yılmaz (2014).

The rapid development of modern data technology has allowed us to have access to large amount of data with possible network structures. On one hand, this poses serious challenges to the analysis of dynamic tail behavior especially within a network composed of a large number of nodes. On the other hand, the data structure bears the opportunity for naturally embedded network information. We take this opportunity of employing network structures, and propose a tail event driven network quantile autoregression model (NQAR), which allows us to make inference based on the underlying processes and to estimate the network dependencies.

In the existing literature, great efforts have been taken to incorporate network information into an econometric framework. For instance, Sewell and Chen (2015) incorporate network information to study the dynamic social behavior of students in a Dutch class by a latent space model. Community detection and extraction methods are studied by Zhao et al. (2011), Amini et al. (2013), and Sewell et al. (2017) using a block structure. Comparably, our proposed framework is related to the recent autoregressive models in large-scale social networks. Based on the cited literature, it is assumed that the behavioural patterns of network users are related to their connected friends (Zhang and Chen, 2013; Zhu et al., 2017). Estimation and computation issues for this situation are intensively discussed (Zhou et al., 2017; Zhu et al., 2018).

A recent paper by Zhu et al. (2017) on network autoregression provide a modelling framework at the mean level. In this work, we extend this to a network quantile autoregression (NQAR) model in order to study conditional quantiles in complex financial networks. Consider a network of firms, connected by their shareholder relationships. Specifically, a nodal connection between two firms can be established if they share major common shareholders. In this context, it makes sense to assume that the conditional quantile function of the response variable (e.g., volatility of stock returns for the firms) is related to underlying exogenous factors. These may include nodal specific variables (e.g., firm specific variables), the lagged response of the same node (e.g., volatility of the same stock in the previous time point), and the lagged responses of other connected nodes. To estimate the parameter, a minimum contrast method is introduced, which is applied to a large-scale network. The corresponding asymptotic properties are established, where the conditions on network structures are given and discussed. Moreover, the stationarity of the NQAR model is investigated, and an impulse analysis under the NQAR model is discussed. Empirically, we discover strong asymmetric network effects of shocks at different quantile levels of stock volatilities in the Chinese financial market. Namely, the network dependence among the absolute returns becomes stronger at the tail level, while at normal times it is not significant.

Finally, our paper is closely related to the recent emerging literature on modeling tail dependence in a complex financial system. Examples include the quantile LASSO framework discussed by Hautsch et al. (2014) and Härdle et al. (2016), where the network relationship is estimated among the financial institutions by imposing an L_1-penalty. Their estimation framework considers a more flexible network formation at the cost of slower convergence rate, as it is in nature a nonparametric estimation. Furthermore, there is also a literature on the tail dependence in multi-dimensional dynamic settings. For example, Cappiello et al. (2014) develop an econometric measure for the comovement of quantiles. In addition, White et al. (2015) provide a very innovative vector autoregressive model for the dynamics of quantiles. Chavleishvili and Manganelli (2016) propose a framework related to the CaViaR process to identify the structural quantile shocks. Comparably, our approach is different mainly in the following three aspects. First, the proposed NQAR model embeds the observed network structure, which provides a parametric estimation framework. Second, it admits trackable quantile dynamics, which facilitates to conduct stationary and impulse response analysis. Third, the model allows for modeling a large number of nodes (a high dimensional setup), and controlling for the observed nodal heterogeneity.

Lastly, we summarize our contribution as follows. Firstly, we provide a novel network quantile model that characterizes the dynamic quantile behavior, which incorporates valuable network information from data. Secondly, we give new definitions of tail-event driven impulse functions under this innovative modeling framework. Thirdly, the asymptotic theories are derived for both the underlying process and estimated parameters. The model stationarity is discussed with insights on its relationship with the given network information. Moreover, detailed conditions on the network structures are derived to ensure the consistency and asymptotic normality of the estimator.

The rest of the paper is organized as follows. Section 2 introduces the network quantile autoregression model and its stationarity properties. Section 3 proposes a novel impulse analysis framework for the network quantile autoregression model. The parameter estimation method is given in Section 4, where the asymptotic properties are presented. An empirical analysis for stocks in Chinese financial markets are conducted in Section 5. Lastly, a conclusion is discussed in Section 6. Extensive numerical studies and all technical details are delegated to the supplementary material.

2. NETWORK QUANTILE AUTOREGRESSION

2.1. Model and Notations

Let U_{it} $(1 \leq i \leq N,\ 1 \leq t \leq T)$ be *iid* random variables following the standard uniform distribution on the set of $[0, 1]$. Assume that a q-dimensional random nodal covariate vector $Z_i = (Z_{i1}, \cdots, Z_{iq})^\top \in \mathbb{R}^q$ is collected for each node. To record the network relationship, we define $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ as the adjacency matrix, where $a_{ij} = 1$ if there is an edge from i to j, otherwise $a_{ij} = 0$.

Following the standard conventions, the nodes are assumed to be not self-related (i.e., $a_{ii} = 0$). Motivated by the univariate autoregression quantile model (Koenker and Xiao, 2006), we consider the network quantile autoregression model as

$$Y_{it} = \beta_0(U_{it}) + \sum_{l=1}^{q} Z_{il}\gamma_l(U_{it}) + \beta_1(U_{it})n_i^{-1}\sum_{j=1}^{N} a_{ij}Y_{j(t-1)} + \beta_2(U_{it})Y_{i(t-1)}, \tag{2.1}$$

where β_js $(0 \le j \le 2)$ and γ_ls $(1 \le l \le q)$ are unknown coefficient functions from $(0, 1)$ to $\mathbb{R}^1$, and $n_i = \sum_{j\neq i} a_{ij}$ is the out-degree for the ith node.

Importantly, the NQAR model (2.1) induces a convenient form of the conditional quantile function of Y_{it}. Denote $Q_Y(\tau|X)$ as the τth conditional quantile of Y seen as a function of X. By assuming that the right side of (2.1) is monotonically increasing in U_{it}, we can write the conditional quantile function of Y_{it} given $(Z_i, \mathbb{Y}_{t-1})$ as:

$$Q_{Y_{it}}(\tau|Z_i, \mathbb{Y}_{t-1}) = \beta_0(\tau) + \sum_{l=1}^{q} Z_{il}\gamma_l(\tau) + \beta_1(\tau)n_i^{-1}\sum_{j=1}^{N} a_{ij}Y_{j(t-1)} + \beta_2(\tau)Y_{i(t-1)}. \tag{2.2}$$

In (2.2), $\beta_0(\tau) + \sum_{l=1}^{q} Z_{il}\gamma_l(\tau)$ reflects the nodal impact invariant over t, where $\beta_0(\tau)$ is referred to as the *baseline function*. The covariates Z_{il} refer to node-specific variables, like, size, leverage ratio, which are invariant in time. It is assumed that the nodal covariates Z_is are independent of the U_{it}s. Next, the second term $n_i^{-1}\sum_{j=1}^{N} a_{ij}Y_{j(t-1)}$ characterizes the network impact from the connected nodes (e.g., firms with common shareholders) (Zhu et al., 2017). The corresponding coefficient function $\beta_1(\tau)$ is then referred to as the *network function*. Lastly, $Y_{i(t-1)}$ captures the impact from the response of the same node in the previous time point. Accordingly, the coefficient function $\beta_2(\tau)$ is then referred to as the *momentum function*. The model (2.2) is related to the autoregression models in spatial econometrics literature, e.g. Lee (2004); Lee and Yu (2009). Although they share the similarity in the construction of the adjacency matrix A, the modelling interests are different. Specifically, the spatial models mainly characterize the simultaneous spatial effect across spatial locations, while our approach mainly focus on modelling the dynamic patterns of the responses. To better understand the NQAR model (2.1), we have the following remarks.

Remark 1. (Monotonicity) Monotonicity is a frequently discussed issue for the quantile autoregression model. A specific example for the monotonicity of (2.1) to hold is that $\gamma_l(\cdot)$s and $\beta_l(\cdot)$s are all monotone increasing functions, and Z_{il}s, Y_{it}s are positive random variables. In other cases, certain data transformation techniques can be conducted to ensure this assumption; see Koenker and Xiao (2006) and Fan and Fan (2010) for more detailed discussions.

Remark 2. (Comparison with NAR Model) One certainly notes (2.2) to be an extension of the network vector autoregression (NAR) model proposed by Zhu et al. (2017). First of all, the NQAR model has a varying coefficient structure which requires to significantly different tools for deriving theoretical properties. Emphatically, the conditional quantile function in the NAR case is

$$Q_{Y_{it}}(\tau|Z_i, \mathbb{Y}_{t-1}) = \beta_0 + \sum_{l=1}^{q} Z_{il}\gamma_l + \beta_1 n_i^{-1} \sum_{j=1}^{N} a_{ij} Y_{j(t-1)} + \beta_2 Y_{i(t-1)} + c_\tau,$$

where c_τ is the quantile of the error distribution, and all the parameters $\beta_1, \beta_2, \gamma_l$ are not related to τ. In contrast, the NQAR model allows coefficient functions to vary over τ. This makes not only the location of the conditional density of Y_{it} be determined by τ, but also the shape of $Q_{Y_{it}}(\tau|Z_i, \mathbb{Y}_{t-1})$ be τ-related. In practice, this model formulation is of particular interest for financial risk management. Specifically, we discuss the following two scenarios in which NQAR is more powerful than the mean case (i.e., NAR model).

Scenario 1. (Tail Behavior) The NQAR model captures asymmetric dependency between the responses at different quantile levels, especially at tail levels. For instance, to measure the conditional VaR of a firm, one can adopt the stock volatility as Y_{it}s for the ith firm and at $\tau = 0.95$. In this case, an asymmetric pattern indicates whether the financial institutions tend to have closer connections in the upper tail (e.g. when the market exhibits high turbulences) than other levels.

Scenario 2. (Robust Estimation) General vector autoregression models are usually sensitive to outliers, which leads to a serious distortion of the estimation (Abello et al., 2013; Li et al., 2015). Consequently, compared to NAR, NQAR is more robust to outliers since it is established on the quantile framework. Specifically, the robust median estimation can be readily obtained by setting $\tau = 0.5$.

Remark 3. (Heteroskedasticity) Heteroskedasticity is a pervasive phenomenon in complex financial systems. The QNAR model could include a vector autoregression model with heteroskedasticity as a special case. We take for example the classical location shift form as mentioned by Koenker and Hallock (2001) as

$$Y_{it} = b_0 + b_1 \overline{Y}_{i(t-1)} + b_2 Y_{i(t-1)} + \sigma_i \varepsilon_{it}, \tag{2.3}$$

where b_0, b_1, b_2 are constants, $\overline{Y}_{i(t-1)} = n_i^{-1} \sum_j a_{ij} Y_{j(t-1)}$, $\sigma_i = b_0 + b_1 \overline{Y}_{i(t-1)} + b_2 Y_{i(t-1)}$, and ε_{it}s are *iid* random variables with distribution function $F(\cdot)$. One could note that Model (2.3) involves the heteroscedasticity in the innovation term. It can be rewritten into,

$$Y_{it} = b_0(1 + \varepsilon_{it}) + b_1(1 + \varepsilon_{it})\overline{Y}_{i(t-1)} + b_2(1 + \varepsilon_{it})Y_{i(t-1)},$$

Specifically, it is a special case of the NQAR model (2.1) by specifying $\beta_0(U_{it}) = b_0(1+\varepsilon_{it})$, $\beta_1(U_{it}) = b_1(1 + \varepsilon_{it})$, and $\beta_2(U_{it}) = b_2(1 + \varepsilon_{it})$, where $\varepsilon_{it} = F^{-1}(U_{it})$.

Remark 4. (Adjacency Matrix) It should be noted that the adjacency matrix A in (2.1) is allowed to take flexible forms according to specific application scenarios. For example, in social network analysis, the adjacency matrix is defined by the natural following-followee relationship (Chen et al., 2013; Zhou et al., 2017; Zhu et al., 2017). Specifically, $a_{ij} = 1$ if the ith node follows the jth node in the network; otherwise $a_{ij} = 0$. In economic and financial applications, one can take several strategies. For instance, Acemoglu et al. (2012) provide a characterization of intersectorial input-output linkage embedded in a network relationship. Alternatively, the network structures of financial institutions are usually constructed according to their financial fundamentals. Specifically, the industrial background, financial statement, shareholder information are commonly employed for network construction (Zou et al., 2017; Antón et al., 2018; Chen et al., 2018). In spatial econometrics, the adjacency matrix can be related to spatial distances between locations (or even economic distance such as a measure of trade flows, e.g., Novy (2013)), where the weight (i.e., a_{ij}) are usually assumed to be monotone decreasing with distance increasing (Cressie and Wikle, 2015; Lee, 2004). In addition, one could take a further flexible approach to model the adjacency matrix A at the first step according to different statistical models. Particularly, the random graph model (Hoff et al., 2012; Herz, 2015) and statistical tests (Granger et al., 2000) can be applied.

2.2. Vector Formulation of NQAR

Next, we organize the NQAR model in (2.1) into vector forms to facilitate further discussions. Define $\mathbb{Y}_t = (Y_{1t}, \cdots, Y_{Nt})^\top \in \mathbb{R}^N$. Let $\mathbf{B}_{0t} = \big(\beta_0(U_{it}) + \sum_l Z_{il}\gamma_l(U_{it}), 1 \le i \le N\big)^\top \in \mathbb{R}^N$, $\mathbf{B}_{1t} = \operatorname{diag}\{\beta_1(U_{it}), 1 \le i \le N\} \in \mathbb{R}^{N\times N}$, $\mathbf{B}_{2t} = \operatorname{diag}\{\beta_2(U_{it}), 1 \le i \le$

$N\}^\top \in \mathbb{R}^{N\times N}$. One can easily verify that $\Gamma = \mathsf{E}(\mathbf{B}_{0t}) = c_0\mathbf{1}_N \in \mathbb{R}^N$, where $c_0 = b_0 + c_Z$, $b_0 = \int_0^1 \beta_0(u)du$ and $c_Z = \mathsf{E}(Z_1)^\top r$ with $r = \big(\int_0^1 \gamma_l(u)du, 1 \le l \le q\big)^\top \in \mathbb{R}^q$. Then the NQAR model (2.1) can be re-written in a vector form as

$$\mathbb{Y}_t = \Gamma + G_t\mathbb{Y}_{t-1} + V_t, \tag{2.4}$$

where $G_t = \mathbf{B}_{1t}W + \mathbf{B}_{2t} \in \mathbb{R}^{N\times N}$, $W = (w_{ij}) = (n_i^{-1}a_{ij}) \in \mathbb{R}^{N\times N}$ is the row-normalized adjacency matrix, and $V_t = \mathbf{B}_{0t} - \Gamma \in \mathbb{R}^N$ is *iid* over t with mean $\mathbf{0}$ and covariance $\Sigma_V = \sigma_V^2 I_N \in \mathbb{R}^{N\times N}$ with $\sigma_V^2 = \sigma_{b_0}^2 + \mathsf{E}\{\gamma^\top(U_{1t})\Sigma_z\gamma(U_{1t})\}$, $\sigma_{b_0}^2 = \int_0^1 \beta_0^2(u)du - \{\int_0^1 \beta_0(u)du\}^2$, and $\Sigma_z = \mathsf{Cov}(Z_1) \in \mathbb{R}^{q\times q}$.

Note that (2.4) is written in the form of a vector autoregression model (Lütkepohl, 2005) with a stochastic coefficient matrix G_t depending on t. It is not hard to see that G_t is linear in the adjacency matrix W. This form borrows the strength of network structure information (i.e., W), and greatly reduces the dimensionality of estimated parameters. For convenience, we discuss the model stationarity based on the vector form (2.4).

2.3. Covariance Stationarity

Given the NQAR model (2.4), it is then of great interest to check the stationarity of $\mathbb{Y}_t$. A process $\{\mathbb{Y}_t\}_{-\infty}^{+\infty}$ is covariance stationary if (a) $\mathsf{E}(\mathbb{Y}_t) = \mu_Y$ for a constant vector $\mu_Y \in \mathbb{R}^N$; (b) $\mathsf{Cov}(\mathbb{Y}_t, \mathbb{Y}_{t-h}) = \mathsf{E}\{(\mathbb{Y}_t - \mu_Y)(\mathbb{Y}_{t-h} - \mu_Y)^\top\} = \Sigma(h)$ with $\Sigma(h) \in \mathbb{R}^{N\times N}$ only related to h. For convenience, let $b_1 = \mathsf{E}\{\beta_1(U_{it})\}$, $b_2 = \mathsf{E}\{\beta_2(U_{it})\}$, $\widetilde{b}_1 = \{\mathsf{E}\,\beta_1^2(U_{it})\}^{1/2}$, $\widetilde{b}_2 = \{\mathsf{E}\,\beta_2^2(U_{it})\}^{1/2}$, $G = \mathsf{E}(G_t) = b_1W + b_2I$, and $G^* = \mathsf{E}(G_t \otimes G_t)$. Then we have the following theorem.

THEOREM 1. *Assume $\widetilde{b}_1 + \widetilde{b}_2 < 1$ and $\mathsf{E}\,|V_{it}| < C$ for some positive constant C. Then the following conclusions hold.*

(a) There exists a unique covariance stationary solution to the NQAR model (2.4) with finite first moment as

$$\mathbb{Y}_t = \sum_{l=0}^{\infty}\Pi_l\Gamma + \sum_{l=0}^{\infty}\Pi_l V_{t-l}, \tag{2.5}$$

where $\Pi_l = \prod_{k=1}^{l} G_{t-k+1}$ for $l \ge 1$ and $\Pi_0 = I$.

(b) Denote $\Sigma_Y = \Sigma(0) = \mathsf{Cov}(\mathbb{Y}_t)$. The stationary mean is $\mu_Y = c_1^{-1}c_0\mathbf{1}_N$ and

$$vec(\Sigma_Y) = (M_1 - c_1^{-2}c_0^2)\mathbf{1}_{N^2} + 2c_1^{-1}c_0(I - G^*)^{-1}vec(\Sigma_{bv}) + (I - G^*)^{-1}vec(\Sigma_V), \tag{2.6}$$

where $c_1 = (1 - b_1 - b_2)^{-1}$, $M_1 = c_1^{-1}c_0^2(1 + b_1 + b_2)(I - G^*)^{-1}$, $\Sigma_{bv} = \sigma_{bv}I_N$, and

$\sigma_{bv} = \mathsf{E}[\{\beta_1(U_{it}) + \beta_2(U_{it})\}V_{it}]$. *Moreover, we have* $\Sigma(h) = G^h\Sigma_Y$ *for any integer* $h > 0$, *and* $\Sigma(h) = \Sigma_Y(G^\top)^{-h}$ *for* $h < 0$.

The proof of Theorem 1 is given in the supplementary material (Appendix A.1). It provides the unique covariance stationary solution of (2.5).

Remark 5. It is straightforward to verify $\widetilde{b}_1 = (b_1^2 + \sigma_{b_1}^2)^{1/2}$, where $\sigma_{b_1}^2 = \mathsf{Var}\{\beta_1(U_{it})\}$. Similarly one can define $\sigma_{b_2}^2 = \mathsf{Var}\{\beta_2(U_{it})\}$ and $\widetilde{b}_2 = (b_2^2 + \sigma_{b_2}^2)^{1/2}$. Therefore the stationarity assumption in Theorem 1 essentially sets constraints on the expectation and variance of the network and momentum functions (i.e., $\beta_1(\cdot)$ and $\beta_2(\cdot)$). It is noteworthy that the assumption does not require $|\beta_1(\tau)| + |\beta_2(\tau)|$ to be strictly less than one for all $\tau \in (0, 1)$. Instead, it imposes an upper bound in L_2-norm, which allows for some "explosive" cases at a specific quantile (i.e., $|\beta_1(\tau)| + |\beta_2(\tau)| > 1$ for some τ). Particularly, if the network and momentum functions are constants, i.e., $\beta_1(\tau) = b_1'$ and $\beta_2(\tau) = b_2'$ (for some constants b_1' and b_2'), the stationarity assumption reduces to $|b_1'| + |b_2'| < 1$, which corroborates to the stationary condition in the mean case (Zhu et al., 2017).

Remark 6. Let us look at the stationary mean μ_Y and covariance Σ_Y. First, note that $\mu_Y = c_1^{-1}c_0\mathbf{1}_N$, thus the stationary mean is the same for all the nodes, and unrelated to the network structure. By contrast, the analytical form for the covariance Σ_Y is more complicated. To better understand how Σ_Y is affected by the network structure, we approximate Σ_Y in the case that β_1 is small (Chen et al., 2013; Zhou et al., 2017; Zhu et al., 2017), namely, $\widetilde{b}_1 = o(1)$. For convenience, define $\widetilde{b}_{12} = \mathsf{E}\{\beta_1(U_{it})\beta_2(U_{it})\}$, $\widetilde{b}_{01} = \mathsf{E}\{\beta_1(U_{it})V_{it}\}$, and $\widetilde{b}_{02} = \mathsf{E}\{\beta_2(U_{it})V_{it}\}$. Employing the Taylor's expansion, Σ_Y can be approximated by

$$\mathsf{Var}(Y_{it}) \approx c_{b1}c_0^2 + \frac{1}{1-\widetilde{b}_2^2}\Big[2(1-b_2)^{-2}\{(1-b_2)\sigma_{bv} + b_1\widetilde{b}_{02}\}c_0 + \sigma_V^2\Big], \tag{2.7}$$

$$\mathsf{Cov}(Y_{i_1t}, Y_{i_2t}) \approx c_{b2}c_0^2 + \frac{1}{(1-b_2^2)^2}\Big\{2(1-b_2)^{-1}\widetilde{b}_{02}c_0 + \sigma_V^2\Big\}\Big\{b_1b_2(w_{i_1i_2} + w_{i_2i_1})\Big\}, \tag{2.8}$$

where $c_{b1} = [(1-\widetilde{b}_2^2)^{-1}\{1-b_2^2+2b_1+2(1-\widetilde{b}_2^2)^{-1}(1-b_2^2)\widetilde{b}_{12}\}-(1-b_2)^{-1}(1-b_2+2b_1)](1-b_2)^{-2}$ and $c_{b2} = (1-b_2)^{-2}(1-b_2^2)^{-2}(1-b_2^2+2b_1+2b_1b_2) - (1-b_2)^{-3}(1-b_2+2b_1)$. Detailed verifications of (2.7) and (2.8) are given in the supplementary material (Appendix A.2). It is worth noting that the variance of Y_{it} is mainly determined by the momentum function $\beta_2(\cdot)$ and the baseline function $\beta_0(\cdot)$. Particularly, a larger $\widetilde{b}_2$ will result in higher variance of Y_{it}. Next, the covariance between Y_{i_1t} and Y_{i_2t} is not only related to $\beta_2(\cdot)$, but is also related to the network function $\beta_1(\cdot)$. Nodes have a higher correlation with each other if b_1

is larger. Moreover, note that $w_{i_1i_2}+w_{i_2i_1} = n_{i_1}^{-1}a_{i_1i_2}+n_{i_2}^{-1}a_{i_2i_1}$. Therefore, the correlation between node i_1 and i_2 is higher if (a) they connect to each other in the network (i.e., $a_{i_1i_2} = a_{i_2i_1} = 1$) and (b) they both have small out-degrees (i.e., small n_{i_1} and n_{i_2}).

2.4. Asymptotic Stationary Distribution

Given the established covariance stationarity, it is then natural to derive the asymptotic stationary distribution. We focus on the long run average of $\mathbb{Y}_t$, namely, $\overline{\mathbb{Y}}_T = T^{-1}\sum_{t=1}^{T}\mathbb{Y}_t$. It reflects the average performance of $\mathbb{Y}_t$ over the whole time period T, and its asymptotic properties are going to be investigated as $T\to\infty$. In this regard, two types of asymptotics exist. The first type is fixed N asymptotics, and the second one is $N\to\infty$ asymptotics. In the following theorem, we first give the result of fixed N asymptotics.

THEOREM 2. *Assume $c_\beta < 1$ and $\mathsf{E}(|V_{it}|^4) < M$, where $c_\beta = \|\beta_1\|_4 + \|\beta_2\|_4$ with $\|\beta_j\|_4 = \mathsf{E}\{\beta_j(U_{it})^4\}^{1/4}$ $(j = 1, 2)$, and M is a finite positive constant. Then the average of $\mathbb{Y}_t$ converges in law to a normal distribution,*

$$\sqrt{T}(\overline{\mathbb{Y}}_T - \mu_Y\mathbf{1}) \xrightarrow{\mathcal{L}} \mathrm{N}(0, \Sigma_Y^*) \qquad as\ \ T\to\infty, \tag{2.9}$$

where $\Sigma_Y^* = G(I-G)^{-1}\Sigma_Y + \Sigma_Y(I-G^\top)^{-1}$.

The proof of Theorem 2 is deferred to the supplementary material (Appendix A.3). Via (2.9), the asymptotic normality of $\sqrt{T}(\overline{\mathbb{Y}}_T - \mu_Y\mathbf{1})$ is provided. One could see that the corresponding asymptotic covariance is equal to the long run covariance $\Sigma_Y^* = \sum_{h=-\infty}^{\infty}\Sigma(h) = (\sigma_{ij}^*) \in \mathbb{R}^{N\times N}$.

Note that the Theorem 2 is established for a fixed N. However, one might consider to extend the result directly to the case $N\to\infty$. On general grounds, this can be difficult, since the convergence in distribution in high dimensions is not well defined. As one possible solution, we discuss the problem under the framework of Gaussian approximation theory, which is formulated by Zhang and Cheng (2014) and Zhang and Wu (2015) for time series analysis. Before we introduce this $N\to\infty$ asymptotics, we first give definition of a convenient distance between two high dimensional vectors. Specifically, the Kolmogorov distance is employed and defined as follows.

DEFINITION 1. *Let $X = (X_1,\cdots,X_N)^\top \in \mathbb{R}^N$, $Y = (Y_1,\cdots,Y_N)^\top \in \mathbb{R}^N$ be N-*

dimensional random vectors. The Kolmogorov distance between X and Y is defined as

$$\rho(X,Y)=\sup_{t\in\mathbb{R}}\Big|P(\|X\|_\infty\le t)-P(\|Y\|_\infty\le t)\Big|,$$

where $\|X\|_\infty$ denotes $\max_{1\le i\le N}|X_i|$ for any arbitrary vector X.

The Kolmogorov distance can be seen as a distance between two distribution functions. Using it, we are able to quantify the distance between $\sqrt{T}(\overline{\mathbb{Y}}_T-\mu_Y\mathbf{1})$ and a given Gaussian random vector. Specifically, define $\widetilde{\mathbb{Y}}_t=\mathbb{Y}_t-\mu_Y$, then we have $\mathsf{Cov}(\widetilde{\mathbb{Y}}_t,\widetilde{\mathbb{Y}}_{t-h})=\Sigma(h)$. Accordingly, let $Z\in\mathbb{R}^N$ be an N-dimensional Gaussian random vector with covariance equal to the long run covariance of $\widetilde{\mathbb{Y}}_t$ as Σ_Y^* (defined in Theorem 2). For a finite sample, the long run covariance is usually approximated by $\Sigma_Y^{*(T)}=\sum_{h=-(T-1)}^{T-1}T^{-1}(T-|h|)\Sigma(h)$. We then have the following.

THEOREM 3. *Assume the same conditions as in Theorem 2. Further assume $\lambda_{\min}(\Sigma_Y^{*(T)})\ge\tau$ for a positive constant τ. In addition, $N=\mathcal{O}\{\exp(T^\delta)\}$ for $0\le\delta<1/11$. Then as $T\to\infty$, we have*

$$\rho\Big(T^{-1/2}\sum_{t=1}^{T}\widetilde{\mathbb{Y}}_t,Z\Big)\to 0. \tag{2.10}$$

The result (2.10) can be seen as an analogue of the central limit theorem in a high dimensional version. It should be noted that to guarantee the $\sqrt{T}$ convergence rate of $T^{-1}\sum_{t=1}^{T}\widetilde{\mathbb{Y}}_t$, the network size N is required to expand in a rate not faster than $\exp(T^\delta)$; see Appendix A.4 for more proof details.

3. PARAMETER ESTIMATION

In this section, we provide an estimation method to the NQAR model (2.1). The asymptotic properties are also established. Let $\theta(\tau)=[\beta_0(\tau),\gamma^\top(\tau),\beta_1(\tau),\beta_2(\tau)]^\top\in\mathbb{R}^{q+3}$ be the parameter vector. In addition, define $X_{it}=(1,Z_i^\top,n_i^{-1}\sum_{j=1}^{N}a_{ij}Y_{jt},Y_{it})^\top\in\mathbb{R}^{q+3}$. Then $\theta(\tau)$ is estimated by

$$\widehat{\theta}(\tau)=\arg\min_\theta\sum_{i=1}^{N}\sum_{t=1}^{T}\rho_\tau\Big\{Y_{it}-X_{i(t-1)}^\top\theta(\tau)\Big\}, \tag{3.1}$$

where $\rho_\tau(u)=u\{\tau-\mathbf{1}(u<0)\}$ is the contrast (check) function for quantile regression. Note that the estimation problem given by (3.1) is equivalent to estimating the quantile regression problem, where Y_{it} is the response variable and X_{it} is the explanatory variable. Consequently, the standard algorithms to estimate the quantile regression model (e.g., simplex methods or interior point methods) can be employed. With regards to the computational perspective, we refer to the Chapter 6 of Koenker (2005) for more details.

Let the conditional density function of Y_{it} given $\mathcal{F}_{t-1}$ be $f_{i(t-1)}(\cdot)$. To facilitate the study of the asymptotic properties, define $\widehat{\Omega}_0 = (NT)^{-1}\sum_{i=1}^{N}\sum_{t=0}^{T-1} X_{it}X_{it}^\top$ and $\widehat{\Omega}_1(\tau) = (NT)^{-1}\sum_{i=1}^{N}\sum_{t=0}^{T-1} f_{it}\{X_{it}^\top\theta(\tau)\}X_{it}X_{it}^\top$ for $\tau \in (0,1)$. Computationally, $f_{it}\big(X_{it}^\top\theta(\tau)\big)$ is approximated by $\hat{f}_{it}\big(X_{it}^\top\hat{\theta}(\tau)\big) = \{X_{it}^\top(\hat{\theta}(\tau_l) - \hat{\theta}(\tau_{l-1}))\}^{-1}(\tau_l - \tau_{l-1})$ for $\tau \in [\tau_{l-1}, \tau_l]$, where $\{\tau_l\}$ is a chosen grid. Next, to prove the asymptotic properties of the estimated parameters, the following assumptions are required.

(C1) (Moment Assumption) Assume $c_\beta < 1$, where c_β is defined in Theorem 2. Further, assume that Z_is are *iid* random vectors, with mean 0, covariance $\Sigma_z \in \mathbb{R}^{q\times q}$ and finite fourth order moment. The same assumption is needed for V_{it}, $1 \le i \le N$ and $0 \le t \le T$. Lastly, assume that $\{Z_i\}$ and $\{U_{it}\}$ to be mutually independent.

(C2) (Network Structure)

(C2.1) (Connectivity) Treat W as the transition probability matrix of a Markov chain, whose state space is defined as the set of all the nodes $\{1, \cdots, N\}$. Assume the Markov chain to be irreducible and aperiodic. In addition, define $\pi = (\pi_1, \cdots, \pi_N)^\top \in \mathbb{R}^N$ to be the stationary distribution vector of the Markov chain (i.e., $\pi_i \ge 0$, $\sum_i \pi_i = 1$, and $W^\top\pi = \pi$). It is assumed that $\sum_{i=1}^N \pi_i^2 \to 0$ as $N \to \infty$.

(C2.2) (Uniformity) Assume $|\lambda_1(W^*)| = \mathcal{O}(\log N)$, where W^* is the symmetric matrix as $W^* = W + W^\top$.

(C2.3) (Convergence) Assume that the following limits exist and finite: $\kappa_1 = \lim_{N\to\infty} N^{-1}\mathrm{tr}(\Sigma_Y)$, $\kappa_2 = \lim_{N\to\infty} N^{-1}\mathrm{tr}(W\Sigma_Y)$, $\kappa_3 = \lim_{N\to\infty} N^{-1}\mathrm{tr}(W\Sigma_Y W^\top)$, $\kappa_4 = \lim_{N\to\infty} N^{-1}\mathrm{tr}\{(I-G)^{-1}\}$, and $\kappa_5 = \lim_{N\to\infty} N^{-1}\mathrm{tr}\{W(I-G)^{-1}\}$.

(C3) (Eigenvalue-bound) Let $\widehat{\Omega}_1(\tau) \to_p \Omega_1(\tau)$ as $\min\{N, T\} \to \infty$ for any $\tau \in (0,1)$, where $\Omega_1(\tau) \in \mathbb{R}^{N\times N}$ is a positive definite matrix. Moreover, there exists positive constants $0 < c_1 < c_2 < \infty$ such that $c_1 \le \lambda_{\min}(\Omega_1(\tau)) \le \lambda_{\max}(\Omega_1(\tau)) \le c_2$ for any $\tau \in (0,1)$.

(C4) (Monotonicity) It is assumed that $X_{it}^\top\theta(\tau)$ $(1 \le i \le N, 1 \le t \le T)$ is a monotone increasing function with respect to $\tau \in (0,1)$.

To gain insights into the conditions, we comment as follows. Condition (C1) is standard conditions on the noise term V_{it}s, nodal covariates Z_is and $\beta(U_{it})$s for the parameter consistency results. This condition can be relaxed to allow for the weak dependence or mixing case over time as long as the asymptotic normality still holds. Condition (C2) is set for the network structure. Specifically, condition (C2.1) ensures the connectivity on the network structure. It implies that all nodes in the network could connect each other within a finite number of steps. This condition can be supported by the empirical phenomenon named as "six degrees of separation" (Newman et al., 2011). Condition (C2.2) assures that the network has certain uniformity properties, i.e. the divergence rate of $\lambda_1(W^*)$ should be of the same rate or slower than $\log(N)$. Consider a fully connected network for example, it can be verified in such a case $\lambda_1(W^*)$ is of order $\mathcal{O}(1)$, which satisfies the condition perfectly. However, this assumption might be violated if huge heterogeneity occurs among nodes (e.g., a "super star" shaped network). Condition (C2.3) states the convergence assumption, which are the values related to network structure. To better understand the condition, we take κ_1 for illustration proposes. It can be written as $N^{-1}\operatorname{tr}(\Sigma_Y) = N^{-1}\sum_i \Sigma_{Y,ii}$, where $\Sigma_{Y,ii}$ is the variance of node i. Consequently, this assumption is satisfied if the average variations of all the nodes in the network converge to a finite constant. Subsequently, condition (C3) assures that the law of large number assumption holds for $\widehat{\Omega}_1(\tau)$. Moreover, the condition guarantees that eigenvalues of the asymptotic covariance matrix in Theorem 4 are bounded from above and below for any $\tau \in B$. Lastly the monotonicity assumption is imposed by condition (C4) to ensure the validity of the quantile regression. Given the conditions, we provide a theorem named as *Network Bahadur Representation*, which leads to the consistency of the parameter estimation.

THEOREM 4 (Network Bahadur Representation). *Under conditions (C1)–(C4), the following representation holds uniformly over $\tau \in B$ (i.e., B is a compact set in $(0,1)$),*

$$\widehat{\theta}(\tau) - \theta(\tau) = (NT)^{-1}\Omega_1(\tau)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} X_{it}\psi_\tau(V_{it\tau}) + r_{NT}(\tau), \tag{3.2}$$

where $\psi_\tau(u) = \tau - I(u<0)$, $V_{it\tau} = Y_{it} - X_{i(t-1)}^\top\theta(\tau)$, and The remainder term satisfies $sup_{\tau\in B}|r_{NT}(\tau)| = o_p\{(NT)^{-1/2}\}$. Therefore, we have $\widehat{\theta}(\tau) \xrightarrow{p} \theta(\tau)$ uniformly for $\tau \in B$ as $\min\{N,T\} \to \infty$.

The proof of Theorem 4 is given in the supplementary material (Appendix B.2). With the consistency of the parameters, we may now present the asymptotic distribution of the estimated parameters.

THEOREM 5. *Under conditions (C1)–(C4), we have*

$$\sqrt{NT}\Sigma_\theta^{-1/2}(\tau)\big\{\widehat{\theta}(\tau)-\theta(\tau)\big\} \xrightarrow{\mathcal{L}} B_{q+3}(\tau)$$

as $\min\{N,T\}\to\infty$*, where* $\Sigma_\theta(\tau)=\Omega_1^{-1}(\tau)\Omega_0\Omega_1^{-1}(\tau)$ *with*

$$\Omega_0=\begin{pmatrix} 1 & \mathbf{0}^\top & c_b & c_b \\ \mathbf{0} & \Sigma_z & \kappa_5\Sigma_z r & \kappa_4\Sigma_z r \\ c_b & \kappa_5 r^\top\Sigma_z & \kappa_3+c_b^2 & \kappa_2+c_b^2 \\ c_b & \kappa_4 r^\top\Sigma_z & \kappa_2+c_b^2 & \kappa_1+c_b^2 \end{pmatrix} \tag{3.3}$$

$c_b=c_1^{-1}c_0$*, and* $B_{q+3}(\tau)$ *is a* $(q+3)$*-dimensional Brownian bridge.*

The proof of Theorem 5 is given in the supplementary material (Appendix B.3). In Theorem 4 and Theorem 5, both N and T are required to diverge to infinity to obtain a $\sqrt{NT}$-consistent result. Although the NQAR model requires that the adjacency matrix should be correctly specified, it is found that the consistency result still holds if the mis-specification amount is under control. See Appendix B.4 in the supplementary materials for a discussion. It is noteworthy that since the nodal covariates Z_i are invariant to time, the minimum requirement is $N\to\infty$ to obtain the consistent estimation of $\gamma(\tau)$.

To better understand the convergence result given in Theorem 5, consider the case that for any fixed τ, $B_{q+3}(\tau)$ reduces to $\mathrm{N}(0,\tau(1-\tau)I_{q+3})$. Specifically, we have the following Corollary on the asymptotic result for fixed τ.

COROLLARY 1. *Under conditions (C1)–(C4), for any fixed* $\tau\in B$ *we have the result* $\sqrt{NT}\big\{\widehat{\theta}(\tau)-\theta(\tau)\big\}\xrightarrow{\mathcal{L}}\mathrm{N}\left(0,\tau(1-\tau)\Sigma_\theta(\tau)\right)$ *as* $\min\{N,T\}\to\infty$*, where* $B\subset(0,1)$ *is a compact set.*

Corollary 1 is a direct implication of Theorem 5. Indeed, by Corollary 1, the asymptotic normality can be obtained at any arbitrary fixed τ. This enables us to conduct pointwise (for any fixed τ) inference on the estimated parameters.

Remark 7. Given the estimated QNAR model, one is interested in measuring goodness-of-fit of the model. A possible solution is based on the approach of Koenker and Machado (1999) who looks at:

$$R^1(\tau)=1-\widehat{Q}(\tau)/\widetilde{Q}(\tau),$$

where $\widehat{Q}(\tau) = \sum_{i=1}^{N} \sum_{t=1}^{T} \rho_\tau\{Y_{it} - X_{i(t-1)}^\top \widehat{\theta}(\tau)\}$ and $\widetilde{Q}(\tau) = \min_{\beta_0} \sum_{i=1}^{N} \sum_{t=1}^{T} \rho_\tau(Y_{it} - \beta_0)$. Similarly, a goodness-of-fit measure can be defined for model prediction, which could also be used for model comparison.

To corroborate the theoretical results, we conduct a number of simulation studies. The details can be found in the Appendix C.1 and C.2 in the supplementary material. In the next section, we discuss an important aspect of the NQAR model: impulse analysis.

4. IMPULSE ANALYSIS

Given the NQAR structure (2.1) it is vital to check marginal influence via an impulse analysis: how does a node in the network react to an exogenous shock imposed on the other nodes at different quantile levels? Particularly, consider a stimulus $\Delta = (\delta_1, \cdots, \delta_N)^\top \in \mathbb{R}^N$ imposed on V_t, i.e. shock to $V_t + \Delta$. Note here we do not consider the structural shock analysis to facilitate a simple discussion.

Then, the response for the ith node at time point t (i.e., Y_{it}) will grow to $Y_{it} + \delta_i$. Following the NQAR model (2.4), the response at time point $(t+l)$, $l \geq 1$ (i.e., $\mathbb{Y}_{t+l}$) is increased by

$$\mathrm{IE}_t(l) = \prod_{k=0}^{l-1} G_{t+l-k} \Delta, \tag{4.1}$$

where $\mathrm{IE}_t(l)$ refers to the *impulse effect* from time t to $t+l$. For instance, if $\Delta = (1, 0, \cdots, 0)^\top$, then the $\mathrm{IE}_t(l)$ is the first column of $\prod_{k=0}^{l-1} G_{t+l-k}$. Note in the a standard impulse analysis of VAR model, the autoregression matrix G_{t+l-k} is a constant matrix. Take the NAR model proposed by Zhu et al. (2017) for example, i.e., $\mathbb{Y}_t = \beta_0 + \beta_1 W \mathbb{Y}_{t-1} + \beta_2 \mathbb{Y}_{t-1} + Z_i^\top \gamma + \mathcal{E}_t$, where all the coefficients take constant forms. Immediately one could obtain the autoregression matrix $G = \beta_1 W + \beta_2 I$. However, in the NQAR model, the autoregression matrix G_t is a stochastic matrix related to $\{U_{it} : 1 \leq i \leq N\}$. As a result, $\mathrm{IE}_t(l)$ cannot be directly evaluated as it is a random process. Therefore, we propose various impulse effects at any tail level in a tractable way.

4.1. Measurements of Impulse Effect

Before we go into the details, we discuss a straightforward way to measure the impulse effect, which can be referred to as average impulse effect (AIE). Naturally, the AIE is directly defined as the expectation of $\mathrm{IE}_t(l)$ as $\mathsf{E}(\mathrm{IE}_t(l)) = G^l \Delta = (b_1 W + b_2 I_N)^l \Delta$. Specifically, the AIE is only related to the average network (b_1) and momentum effect

(b_2). It is noteworthy that the AIE is no longer related to t but only depends on the time lag l. It can be further derived $|\mathbf{1}^\top \mathsf{E}(\mathrm{IE}_t(l))| \leq N(\widetilde{b}_1 + \widetilde{b}_2)^l C_\Delta$, where $C_\Delta = \max_i |\delta_i|$. Therefore, it can be concluded that the AIE will reduce to 0 as $l \to \infty$, if the stationary condition in Theorem 1 is satisfied. Although the AIE is easy to understand, it could only measure the average level of impulse effect. As an extension, we propose the following three quantile specific measurements to measure the impulse effect.

Type I. (Interval Impulse Effect) It can be noted that the AIE can characterize the impulse effect on average. However, it cannot capture asymmetric effects between different tail levels. To this end, we define the interval impulse effect (IIE) from t to $t+l$ within the interval $[\tau_1, \tau_2]$, $(0 < \tau_1 < \tau_2 < 1)$ as

$$\begin{aligned}
\mathrm{IIE}_{\tau_1\tau_2}(l) &= \mathsf{E}\Big\{\prod_{k=0}^{l-1} G_{t+l-k}\Delta \Big| U_{im} \in [\tau_1, \tau_2], 1 \leq i \leq N, t+1 \leq m \leq t+l\Big\} \\
&= (c_{\beta_1,\tau_1\tau_2} W + c_{\beta_2,\tau_1\tau_2} I_N)^l \Delta,
\end{aligned}$$

where $c_{\beta_1,\tau_1\tau_2} = \int_{\tau_1}^{\tau_2} \beta_1(u)du$ and $c_{\beta_2,\tau_1\tau_2} = \int_{\tau_1}^{\tau_2} \beta_2(u)du$. As one can see, the size of IIE is determined by the integration of $\beta_1(u)$ and $\beta_2(u)$ within any selected $[\tau_1, \tau_2]$. For example, to measure the effects in the upper tail, at the middle level and in the lower tail, one can split $(0,1)$ equally into three intervals (i.e., $(0, 1/3)$, $[1/3, 2/3)$, $[2/3, 1)$) and compare the IIEs for different intervals respectively.

Type II. (Impulse Effect Intensity) IIE can distinguish effects at different quantiles. However, due to the unknown function forms of β_1 and β_2, the integration can still be hard to compute. On the other hand, note that the IIE can be defined in any interval in $(0,1)$. Motivated by this, we consider a sufficiently small interval $[\tau, \tau+\delta]$, and define the impulse effect intensity (IEI) at τ as

$$\begin{aligned}
\mathrm{IEI}_\tau(l) &= \lim_{\delta \to 0} \delta^{-l}\, \mathsf{E}\Big(\prod_{k=0}^{l-1} G_{t+l-k}\Delta \Big| U_{im} \in [\tau, \tau+\delta], 1 \leq i \leq N, t+1 \leq m \leq t+l\Big) \\
&= \big\{\beta_1(\tau)W + \beta_2(\tau)I_N\big\}^l \Delta,
\end{aligned}$$

where $\beta_1(u)$ and $\beta_2(u)$ are assumed to be continuous at τ. By definition, $\mathrm{IEI}_{l,\tau}$ can reflect the impulse impact at the τth quantile, and is easy to compute as long as the estimates of $\beta_1(\tau)$ and $\beta_2(\tau)$ are obtained.

Type III. (Pseudo Quantile Impulse Response Function) Similar to White et al. (2015), we can define the pseudo impulse response function. Recall that we impose a stimulus on $\mathbb{Y}_t$ and turn it to $\mathbb{Y}_t + \Delta$, and the l-step ahead the impulse effect $\mathrm{IE}_t(l)$ is given by (4.1). We are interested in checking the change to the conditional quantile $Q_{Y_{i(t+l)}}(\tau|\mathbb{Y}_{t+l-1}, Z_i)$, which is the pseudo quantile impulse response function according to White et al. (2015). From (2.2), we have

$$Q_{Y_{i(t+l)}}(\tau|Z_i, \mathbb{Y}_{t+l-1}) = \beta_0(\tau) + \sum_{l=1}^{q} Z_{il}\gamma_l(\tau) + \beta_1(\tau)\sum_{j=1}^{N} w_{ij}Y_{j(t+l-1)} + \beta_2(\tau)Y_{i(t+l-1)}.$$

Therefore, the pseudo quantile impulse response function is given by

$$Q_{\widetilde{Y}_{i(t+l)}}(\tau|Z_i, \widetilde{\mathbb{Y}}_{t+l-1}) - Q_{Y_{i(t+l)}}(\tau|Z_i, \mathbb{Y}_{t+l-1}) = \beta_1(\tau)\sum_{j=1}^{N} w_{ij}\mathrm{IE}_{t,j}(l-1) + \beta_2(\tau)\mathrm{IE}_{t,i}(l-1),$$

where $\mathrm{IE}_{t,j}(l-1)$ is the jth element of $\mathrm{IE}_t(l-1)$. Due to the randomness of $\mathrm{IE}_t(l-1)$, the pseudo quantile impulse response function can be measured by the above two methods.

Given the different types of impulse effect measurement, a cross-sectional impulse analysis can be conducted. Assume that one unit stimulus is imposed on the ith node, a cross-sectional impulse analysis aims to analyzing its impact on the other nodes. For instance, the impulse analysis can be critical in a network of financial institutions. It delivers an important message on the systemic risk spillover of an institution. Take the IEI as an example and assume $\Delta = (\delta_i)^\top$ with only $\delta_i = 1$ and $\delta_{i'} = 0$ (for all $i' \neq i$). The IEI from node i to j can be defined by the jth element of $\mathrm{IEI}_\tau(l)$, which is then denoted as $\mathrm{IEI}_\tau^{(i,j)}(l)$. Equivalently, $\mathrm{IEI}_\tau^{(i,j)}(l)$ is equal to the (j,i)th element of the matrix $\big\{\beta_1(\tau)W + \beta_2(\tau)I_N\big\}^l$. If $\mathrm{IEI}_\tau^{(i,j)}(l)$ is sufficiently large and decays slowly as $l \to \infty$, the jth node (e.g. risk factor) can be seriously affected by the shock on the ith node for a long time.

4.2. Influential Node Analysis

The impulse effect measures the impact from t to $t+l$ given a stimulus $\Delta = (\delta_i)^\top$. Specifically, consider that one unit stimulus is imposed on the node i with $\delta_i = 1$ and $\delta_j = 0$ for all $j \neq i$. As a result, the impulse impact of that unit stimulus could be measured with respect to a particular node i. Such amount of influence could reflect the influential power of the node, which leads to a quantification of influential nodes. Empirically, the influential nodes in a complex financial system should be paid particular attention with financial regulation.

To facilitate the analysis, first define the total network average impulse effect (T-NAIE) as the summation of the cumulated AIE as $\mathrm{TNAIE}(\Delta) = \sum_{l=0}^{\infty} \mathbf{1}^\top \mathsf{E}(\mathrm{IE}_t(l)) = \sum_{l=0}^{\infty} \mathbf{1}^\top G^l \Delta = \mathbf{1}^\top (I - G)^{-1} \Delta$. Note that the definition is given by using the impulse measurement AIE, but it would be similar to use the other impulse measures (IIE and IEI). Further write $\mathrm{TNAIE}(\Delta)$ as $\mathrm{TNAIE}(\Delta) = \sum_{i=1}^{N} \nu_i \delta_i$, where ν_i is the ith element of the vector $\nu = (I - G^\top)^{-1}\mathbf{1}$. Then we have $\mathrm{TNAIE}(\Delta) = \nu_i$ if we set $\delta_i = 1$ and $\delta_j = 0$ if $j \neq i$. It measures the effects of one unit perturbation from the node i on the whole network. We thus define ν_i as the *influential power* of node i.

However, in practice, ν_i can be hard to compute, as the calculation of ν involves the inverse of a high dimensional matrix $(I - G^\top) \in \mathbb{R}^{N\times N}$. Following the idea of Remark 4 of Theorem 1, we can approximate ν_i by the first order Taylor's expansion, which would lead to

$$\nu_i \approx 1/(1 - b_2) + (1 - b_2)^{-2} b_1 \sum_j n_j^{-1} a_{ji}.$$

Suppose $b_1 > 0$, then the influential power of node i is mainly determined by the quantity $\sum_j n_j^{-1} a_{ji}$, which is referred to as the *weighted degree* of the node i. Generally speaking, the weighted degree is an approximation to the influential power of the nodes. Therefore one may rank the nodes' influences based on the weighted degrees. Computationally, the calculation of this weighted degree does not involve complex computation of inverse of a high dimensional matrix $(I - G^\top)$, as well as specific values of b_1 and b_2, but only the network structure information. As a result, nodes with larger weighted degrees tend to be followed (connected) by a large amount of nodes (i.e., $\sum_j a_{ji}$). Moreover, at the same time, the connected nodes should have less out-degrees (i.e., small n_js).

5. FINANCIAL CONTAGION AND SHARED OWNERSHIP

In this section, we study financial risk contagion mechanisms arising from the common shared ownership information. Specifically, we focus on the Chinese Stock Market in 2013. The dataset consists of $N = 2,442$ stocks in the Chinese A share market, which are traded in the Shanghai Stock Exchange and the Shenzhen Stock Exchange. Here $N = 2,442$ is the size of cross section. For each stock, the weekly price is recorded for $T = 52$ weeks. The Y_{it} is the log-transformed weekly absolute return volatility, where the absolute return volatility is calculated as the absolute stock return for $t = 1, \cdots, T$. The average volatility of all stocks at $t = 1, \cdots, T$ is calculated and visualized in the left panel of Figure 1. A relatively higher volatility level can be captured in May and July. To describe the cross sectional information, we average for each stock the volatilities over time. That leads to a median volatility level of 0.014. In addition, we calculate the cross sectional correlations for all the stocks, which leads to $N(N-1)$ correlations. The histogram of cross sectional correlations is given in the right panel of Figure 1, where the mean correlation level is 0.105. This implies on average the stocks tend to be positively correlated.

To construct the network structure, the top ten shareholders' information for each stock are collected, which are referred to as *major* shareholders of the stock. For the ith and jth stock, $a_{ij} = 1$ if they share at least one major common shareholder, otherwise $a_{ij} = 0$. The shareholder network reflects an important information of inter-corporate dependence. Particularly, this is an important research problem of financial risk management. Corsi et al. (2016) argue about the common shareholder effect from the perspective of the diversification cost. They discovered that a reduction of diversification cost will lead to increasing level of diversification and thus increases the degree of overlap (common shareholder). This explains that why financial institutions, who have common shareholders are more likely to be highly correlated. We visualize the network structure among the top 100 stocks ranked by market values in Figure 2. The resulting network density is 3.9%. In addition, we would like to comment that one could construct the adjacency matrix by other approaches (e.g., according to their industrial background) and the proposed method could still be applied.

Besides this shared ownership information, firm specific variables are also taken into consideration. Motivated by Fama and French (2015), consider the following $K = 6$ covariates to represent stocks' fundamentals: SIZE (measured by the logarithm of market value), BM (book to market ratio), PR (increased profit ratio compared to the last year), AR (increased asset ratio compared to the last year), LEV (leverage ratio), and Cash (cash flow of the firm). Eventually all covariates are normalized within the interval $[0, 1]$.

We then proceed with the network analysis using the NQAR model. The results of our NQAR model yields Table 1. Both the estimates and the p-values are reported at quantiles $\tau = 0.05$, 0.30, 0.50, 0.70, and 0.95 (from left to right). One could discover that the stocks have stronger network effect and momentum effect in the upper tail (i.e., $\tau = 0.95$) than the other quantiles. This indicates that stocks tend to have higher dependence through the network when the market is exposed to a higher volatility level. While on normal occasions (e.g., $\tau = 0.5$), the network effect tends to be insignificant. Besides, the size (i.e., CAP), the book to market ratio (BM), and the leverage ratios (LEV) are shown to have negative correlations with the conditional quantile level of the volatility at $\tau = 0.95$

and $\tau = 0.5$. To access the model fitting level, we apply a model diagnosis procedure on residuals by using the QACF (quantile ACF) measure proposed by Li et al. (2015). The details and discussions are given in Appendix C.3 in the supplementary material. There is no strong evidence showing dynamic and cross-sectional dependence remained in the residuals.

Lastly, we include the NAR model (Zhu et al., 2017) for a comparison. The corresponding estimation results are given in Table 2. It can be seen that the network effect (i.e., $\hat{\beta}_1$) is no longer significant with p-value much larger than 10%. That implies the NAR model might not be suitable to detect the asymmetric effects in this volatility autoregression problem. Next, we compare the NQAR model with the LASSO method in multivariate quantile regression QLASSO (The tuning parameter is chosen to maximize $R^1(\tau)$ of (5.1)) see Hautsch et al. (2014). Specifically, we use the time periods with $t = 1, \cdots, m$ for model training and the remaining for prediction. Following Koenker and Machado (1999), we define the prediction goodness-of-fit measure as

$$R^1(\tau) = 1 - \widehat{Q}(\tau)/\widetilde{Q}(\tau), \tag{5.1}$$

where $\widehat{Q}(\tau) = \sum_{i=1}^{N} \sum_{t=m+1}^{T} \rho_\tau\{Y_{it} - X_{i(t-1)}^\top \widehat{\theta}(\tau)\}$ and $\widetilde{Q}(\tau) = \min_{\beta_0} \sum_{i=1}^{N} \sum_{t=m+1}^{T} \rho_\tau(Y_{it} - \beta_0)$. Since for the LASSO method it is required that the number of nodes should be less than the total time periods for feasible estimation, we then randomly select $n = 40$ nodes each time, and keep $m = 46$ for training and the last 5 weeks for prediction. In addition, we set $\tau = 0.95$ to compare the prediction accuracy for the tail event. The experiment is repeated for $R = 500$ times to obtain reliable results. The goodness-of-fit measures are reported in Figure 5. It is evident that the NQAR model has better prediction power than the QLASSO model.

The impulse analysis as discussed in Section 4 is applied for stocks of five well-known banks in China. They include Bank of China (BOC), China Merchants Bank (CMB), Industrial and Commercial Bank of China (ICBC), Ping An Bank (PAB), and Shanghai Pudong Development Bank (SPDB). Specifically, $\mathrm{IEI}_\tau(l)$ ($\tau = 0.05, 0.5, 0.95$) in Section 4.1 is computed within 5 banks at time lags $l = 1, \cdots, 10$, which are plotted in Figure 3. The (i, j)th panel in Figure 3 denotes the impulse impact of the jth bank on the ith bank (i.e., $\mathrm{IEI}_\tau^{(j,i)}(l)$). Significant asymmetric effects across different quantiles can be observed, where larger IEI can be detected in the upper tail ($\tau = 0.95$). Note that the estimated network effect $\beta_1(\tau)$ is very small at $\tau = 0.05$ and $\tau = 0.5$. Therefore, the resulting impulses $\mathrm{IEI}_\tau^{(j,i)}(l)$ is much smaller than the higher quantile level $\tau = 0.95$, which results in an almost flat impulse line in Figure 3. Moreover, it is observed that the mutual impulse impacts between BOC, CMB, and ICBC are much stronger than between the other two banks. Next, to evaluate the nodal influence, the influential node analysis is conducted. By Section 3.2, the influential power can be calculated by $\hat{\nu} = \{(1-\hat{b}_2)I_N - \hat{b}_1 W^\top\}^{-1}\mathbf{1}$, where $\hat{b}_1 = 10^{-1}\sum_{m=0}^{9}\hat{\beta}_1(0.05+0.1m)$ and $\hat{b}_2 = 10^{-1}\sum_{m=0}^{9}\hat{\beta}_2(0.05+0.1m)$ are computed as the numerical approximations for b_1 and b_2. Subsequently the influential power is plotted against the weighted degrees on the right panel of Figure 4, where a strong linear pattern is detected. In addition, the histogram of the weighted degrees is given on the left panel of Figure 4, where a skewed distribution pattern can be noticed. This indicates that a small portion of nodes possesses a large amount of influential power.

6. CONCLUSION

In this article, we propose a network quantile autoregression framework, which models network dynamics in a complex tail event-driven system. The stationarity of the model is discussed by considering the underlying stochastic process. To estimate the NQAR parameters, a minimum contrast estimation is employed. The asymptotic properties of the resulting estimators are investigated, which are closely related to the given network information. We illustrate the performance of the NQAR model via simulation studies and an application in the Chinese stock markets. In particular, significant asymmetric dependency at different levels of quantiles can be detected. Specifically, a stronger network effect can be found between stocks when higher volatility level is exposed to the market. This further confirms the usefulness of our proposed methodology.

We discuss here several potential future research topics. First, we comment on possible extensions of the model forms. First, it could be noted that the responses of the NQAR model are continuous. As a natural extension, a quantitative framework can be established for discrete response variables. Second, it should be noted that the NQAR model focus on lagged dependence. Thus an alternative modelling framework is to consider the contemperaneous correlation of nodes at the same time period, which would lead to the spatial quantile autoregression model. Next, suggested by the empirical evidence, we find that the dynamic patterns could be different during different time periods. See Appendix C.4 in the supplementary materials for discussions. Particularly, one tends to see stronger asymmetric network effects when the market exhibits higher turbulence. Therefore, more flexible model forms can be designed to model this phenomenon. Lastly, heteroskedasticity is a pervasive phenomenon in financial data and should be taken into consideration. Although it has been shown in Remark 3 that the NQAR model could allow for certain types of heteroskedasticity, however, it is still worthwhile to discuss more general settings. In addition, with respect to the model form, one could consider allowing node specific network and momentum functions to reflect more heterogeneity in the network.

Next, the specification of the adjacency matrix should be investigated. It is found that the estimation would be biased when the network relationship is seriously mis-specified. As a complement to the NQAR model, one could consider the approach of Hautsch et al. (2014) to estimate the network relationship among the nodes. In addition to that, bias correction methods should be developed accordingly. Another flexible approach is to make the adjacency matrix to be time varying and related to exogenous covariates. Under that specification, new estimation methods should be discussed.

Thirdly, note that the NQAR model (2.1) requires continuous observations for each subject i. This may not be applicable in some scenarios. For example, the daily stock price will be missing when the stock market is closed on weekends. The model should be further adjusted to allow for missing values in such scenarios.

Lastly, the impulse analysis should be further investigated to allow for possibly structural shocks. Consider the NAR model for example, i.e., $\mathbb{Y}_t = \beta_0 + \beta_1 W \mathbb{Y}_{t-1} + \beta_2 \mathbb{Y}_{t-1} + Z_i^\top \gamma + \mathcal{E}_t$, where the coefficients take constant forms and $\mathsf{Cov}(\mathcal{E}_t) = \Sigma_\varepsilon$. By conducting a Cholesky decomposition on Σ_ε and assuming an empirical causal chain of the nodes on the identification of the structural model, we could have $\Sigma_\varepsilon = LL^\top$. An equivalent structural VAR form is $L^{-1}\mathbb{Y}_t = \beta_0 L^{-1} + L^{-1} G \mathbb{Y}_{t-1} + L^{-1} Z_i^\top \gamma + e_t$, where $G = \beta_1 W + \beta_2 I$ and $e_t = L^{-1}\mathcal{E}_t$. Further it can be transformed to $\mathbb{Y}_t = (I - L^{-1})\mathbb{Y}_t + \beta_0 L^{-1} + L^{-1} G \mathbb{Y}_{t-1} + L^{-1} Z_i^\top \gamma + e_t$. Given this form, one could be able to analyze the instantaneous effect by making an impulse on $\mathbb{Y}_t$. We refer to Lütkepohl (2005) for more discussions. However, in the case of NQAR model, it is not so straightforward. That is because that the N-QAR model has a non-linear model structure. Furthermore, it is assumed the underlying generating process U_{it} is independent with each other. A possible extension is to allow dependency across nodes for U_{it}. That will facilitate the discussion of the structural shock for the NQAR model. However, in such a case, the dependency structure will exist simultaneously with $\{\beta_k(U_{it}), 1 \le i \le N\}$ for $k = 0, 1, 2$. The analytical form of the impulse function would thus not be explicit. Since the extension might be non-trivial, we would like to leave it for future research topics.

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Table 1: The detailed NQAR analysis results for the Chinese Stock dataset (τ =0.05, 0.5, 0.95). The parameter estimates ($\times 10^{-2}$) are reported for $\tau = 0.05, 0.5, 0.95$, where the standard error ($\times 10^{-2}$) is given in parentheses. The p-values are also reported.

	$\tau = 0.05$		$\tau = 0.3$		$\tau = 0.5$		$\tau = 0.7$		$\tau = 0.95$	
	Estimate	p-value	Estimate	p-value	Estimate	p-value	Estimate	p-value	Estimate	p-value
$\hat{\beta}_0$	0.05 (0.00)	< 0.01	0.55 (0.02)	< 0.01	1.00 (0.04)	< 0.01	1.50 (0.05)	< 0.01	2.96 (0.13)	< 0.01
$\hat{\beta}_1$	0.00 (0.01)	0.99	-1.57 (0.60)	< 0.01	-0.04 (0.77)	0.95	2.29 (0.89)	0.01	6.09 (2.16)	< 0.01
$\hat{\beta}_2$	4.16 (0.14)	< 0.01	23.75 (0.39)	< 0.01	35.70 (0.47)	< 0.01	47.18 (0.54)	< 0.01	67.84 (1.13)	< 0.01
SIZE	0.00 (0.01)	0.98	-0.35 (0.07)	< 0.01	-1.00 (0.09)	< 0.01	-1.75 (0.11)	< 0.01	-4.10 (0.28)	< 0.01
BM	0.00 (0.01)	0.99	-0.10 (0.04)	0.02	-0.29 (0.04)	< 0.01	-0.56 (0.08)	< 0.01	-0.71 (0.25)	< 0.01
PR	0.00 (0.00)	1.00	-0.30 (0.02)	< 0.01	-0.30 (0.12)	0.01	-0.11 (0.13)	0.39	0.39 (0.38)	0.31
AR	-0.02 (0.03)	0.55	-0.30 (0.03)	< 0.01	-0.66 (0.11)	< 0.01	-0.24 (0.40)	0.54	-0.47 (0.36)	0.20
CASH	-0.01 (0.00)	0.01	-0.02 (0.05)	0.72	-0.14 (0.06)	0.01	-0.15 (0.08)	0.05	-0.05 (0.27)	0.86
LEV	0.00 (0.01)	0.97	-0.58 (0.05)	< 0.01	-0.79 (0.05)	< 0.01	-1.34 (0.02)	< 0.01	-2.42 (0.44)	< 0.01

Table 2: The NAR analysis results for the Chinese Stock dataset. The parameter estimates ($\times 10^{-2}$) and the standard error ($\times 10^{-2}$) are reported. The p-values are also given.

	Estimate	SE	p-value
BASELINE $\hat{\beta}_0$	1.19	0.04	< 0.01
NETWORK $\hat{\beta}_1$	0.22	0.67	0.74
MOMENTUM $\hat{\beta}_2$	36.79	0.27	< 0.01
SIZE $\hat{\gamma}_1$	-1.60	0.10	< 0.01
BM $\hat{\gamma}_2$	-0.35	0.08	< 0.01
PR $\hat{\gamma}_3$	-0.03	0.10	0.75
AR $\hat{\gamma}_4$	-0.19	0.15	0.19
CASH $\hat{\beta}_0$	0.02	0.11	0.82
LEV $\hat{\beta}_1$	-1.34	0.10	< 0.01

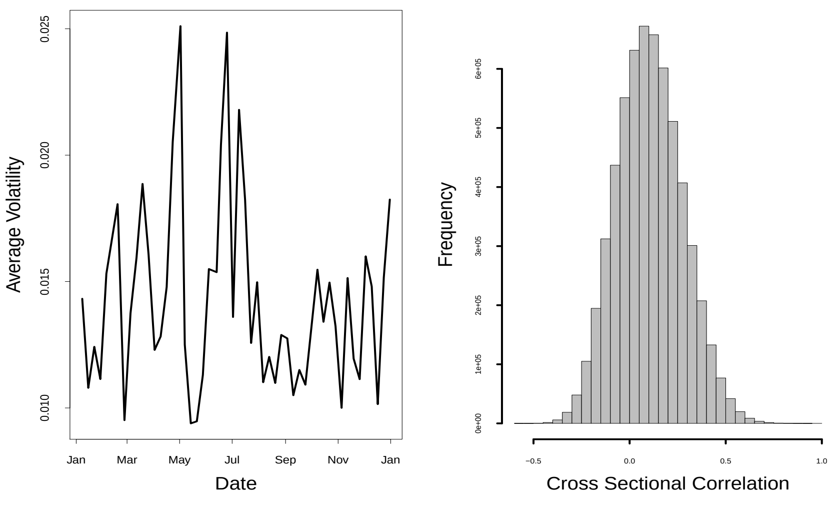

Figure 1: Left panel: the average stock volatility of Chinese A stock market in 2013. Higher volatility level can be captured in the first half of 2013; right panel: histogram of cross sectional correlations for the stocks. The average correlation level is 0.105.

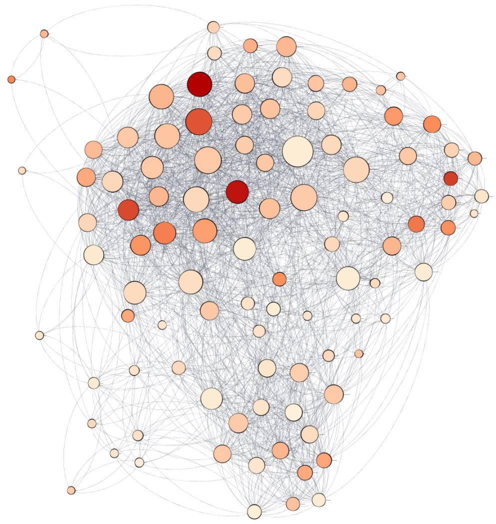

Figure 2: The common shareholder network of top 100 market value stocks in 2013. The larger and darker points imply higher market capitalization.

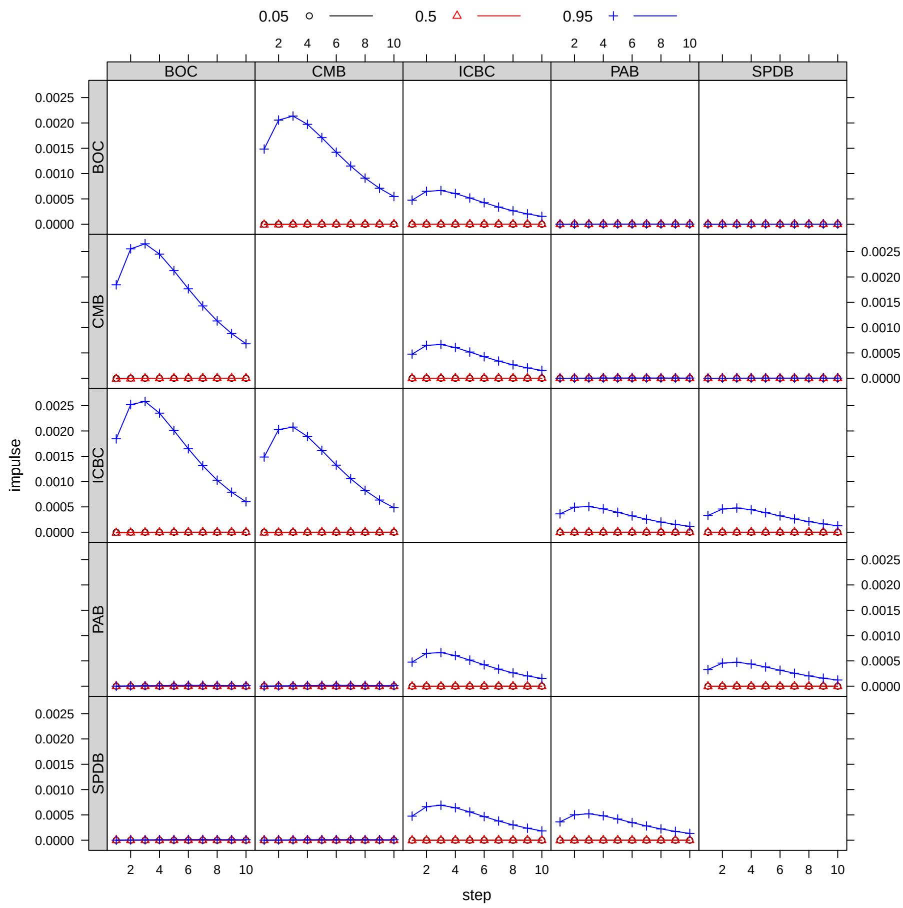

Figure 3: Impulse analysis for $\tau = 0.05, 0.5, 0.95$. The cross-sectional impulse effect intensity between BOC, CMB, ICBC, PAB, and SPDB are given. The (i, j)th panel denotes the impulse impact of the jth bank on the ith bank (i.e., $\mathrm{IEI}_{\tau}^{(j,i)}(l)$).

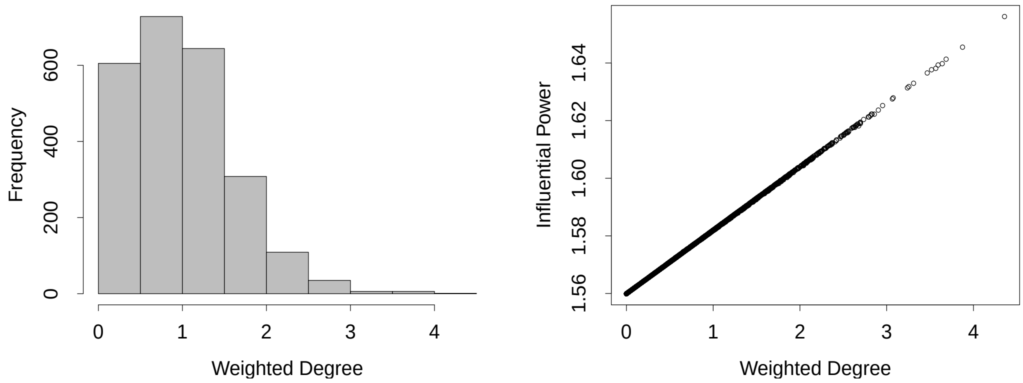

Figure 4: Left panel: the histogram of the weighted degrees; right panel: the influential power against weighted degrees.

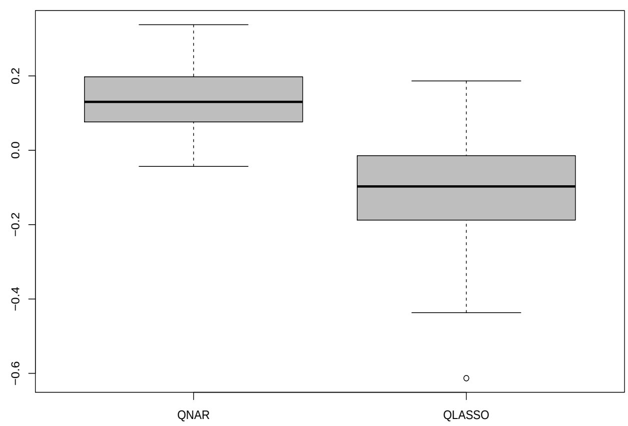

Figure 5: The prediction goodness-of-fit measure $R^1(\tau)$ at $\tau = 0.95$ for the QNAR model and the QLASSO methods. Better prediction performance can be detected for the QNAR model.